Polynomial Growth Harmonic Functions on Finitely Generated Abelian Groups

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Abstract

In the present paper, we develop geometric analysis techniques on Cayley graphs of finitely generated abelian groups to study the polynomial growth harmonic functions. We provide a geometric analysis proof of the classical Heilbronn theorem [22] and the recent Nayar theorem [41] on polynomial growth harmonic functions on lattices $\mathbb{Z}^n$ that does not use a representation formula for harmonic functions. In the abelian group case, by Yau's gradient estimate we actually give a simplified proof of the more general polynomial growth harmonic function theorem of Alexopoulos [3] for groups of polynomial volume growth. We also calculate the precise dimension of the space of polynomial growth harmonic functions on finitely generated abelian groups. While the Cayley graph not only depends on the abelian group, but also on the choice of a generating set, we find that this dimension depends only on the group itself.

1 Introduction

Classically, in 1948 Heilbronn [22] proved the polynomial growth harmonic function theorem on the lattice $\mathbb{Z}^n$ that polynomial growth discrete harmonic functions are polynomials, and calculated the dimension of the space of polynomial growth discrete harmonic functions. Recently, in 2009 Nayar [41] gave another proof of this theorem by the probabilistic method. Their proofs all depend on the representation formula for discrete harmonic functions. In this paper, we give a geometric analysis proof that can yield more general results.

The study of harmonic polynomials on $\mathbb{R}^n$ is classical, and the precise dimension calculation of harmonic polynomials can be found in [33]. In 1975, Yau [50] proved the Liouville theorem for harmonic functions on Riemannian manifolds with nonnegative Ricci curvature. Then Cheng-Yau [8] used Bochner's technique to derive the gradient estimate for positive harmonic functions called Yau's gradient estimate which implies that sublinear growth harmonic functions on these manifolds are constant. Then Yau [51, 52] conjectured that the space of polynomial growth harmonic functions with growth rate less than or equal to d

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on Riemannian manifolds with nonnegative Ricci is of finite dimension. Li-Tam [34] and Donnelly-Fefferman [18] independently solved the conjecture for manifolds of dimension two. Then Colding-Minicozzi [10, 11, 12] gave the affirmative answer by using the volume doubling property and the Poincaré inequality for arbitrary dimension. The simplified argument by the mean value inequality can be found in [32, 13] where the dimension estimate is asymptotically optimal. This inspired many generalizations on manifolds [49, 48, 46, 35, 36, 7, 27, 31] and on singular spaces [16, 28, 24, 25, 26]. In this paper, we give the precise dimension calculation of polynomial growth harmonic functions on finitely generated abelian groups.

In another direction, Avellaneda-Lin [5] first proved the polynomial growth harmonic function theorem for elliptic differential operators with periodic coefficients in $\mathbb{R}^n$ (see [30, 1, 2, 4, 38, 37] for more generalizations). Alexopoulos [3] proved a more general polynomial growth harmonic function theorem for groups of polynomial volume growth which contained our case. By a famous theorem of Gromov [20], every finitely generated group G of polynomial growth is virtually nilpotent (i.e. it has a nilpotent subgroup H of finite index). For some torsion-free subgroup H' of H, it can be embedded as a lattice in a simply connected nilpotent Lie group N. By considering the exponential coordinate of N, N is identified with $\mathbb{R}^n$ (see [42]). Alexopoulos proved that every polynomial growth harmonic function on G is the restriction to H' of some polynomial on $\mathbb{R}^n$. He used the homogenization theory and Krylov-Safonov's argument to compare the discrete heat kernel on H' and the continuous heat kernel for some sub-Laplacian on N. Then he adapted some ideas of Avellaneda-Lin [5] to prove a Taylor formula for harmonic functions on G. Instead of doing that, we shall prove Yau's gradient estimate on abelian groups to simplified the argument in this special case.

We develop the geometric analysis techniques on Cayley graphs of finitely generated abelian groups. Firstly, we prove Yau's gradient estimate (see Theorem 1.1) for positive discrete harmonic functions. Note that Kleiner [28] obtained the Poincaré inequality on Cayley graphs of finitely generated (not necessarily abelian) groups (see also [45]). For the abelian case, combining it with the natural volume doubling property, we obtain the uniform Poincaré inequality (see Lemma 3.1) from which the mean value inequality follows by the Moser iteration. In addition, for an abelian group, Bochner's formula for discrete harmonic functions can be easily calculated, i.e. $|\nabla u|^2$ is subharmonic for the discrete harmonic function u which is essentially due to Chung-Yau [9] and Lin-Yau [40] for Ricci flat graphs. Then these results together imply Yau's gradient estimate on Cayley graphs of finitely generated abelian groups as in the case of Riemannian manifolds with nonnegative Ricci curvature. By the fundamental theorem of finitely generated abelian group (see [47, 43]), any finitely generated abelian group G is isomorphic to the direct sum $\mathbb{Z}^m \bigoplus \oplus_{i=1}^{l} \mathbb{Z}_{q_i}$, where $m, l \in \mathbb{N}$, $q_i = p_i^{a_i}$ for some prime number p_i and $a_i \in \mathbb{N}$.

Theorem 1.1 (Yau's gradient estimate). *Let (G, S) be the Cayley graph of a finitely generated abelian group with symmetric generating set S (i.e. $S = -S$), and $G \cong \mathbb{Z}^m \bigoplus \oplus_{i=1}^{l} \mathbb{Z}_{q_i}$. Then there exist constants C_1 and C_2 depending only on m and S, such that for any $x \in G$, $R \geq 1$ and any positive discrete harmonic*

function f on $B_{C_1R}(x)$, we have

$$|\nabla f|(x) \leq \frac{C_2}{R} f(x). \tag{1.1}$$

Secondly, from Yau's gradient estimate, we know that $|\nabla f| \leq CR^{d-1}$, on B_R for $R \geq R_0$ if the discrete harmonic function f satisfies $|f| \leq CR^d$ on B_R for $R \geq R_1$. That is, the growth order decreases when we take derivatives.

This is the key to an induction argument to give a geometric analysis proof of Heilbronn's theorem. This scheme will, in fact, work for all abelian groups. Let us denote the set of polynomial growth harmonic functions of growth rate less than or equal to d on the Cayley graph (G, S) by $H^d(G,S) := \{u : G \to \mathbb{R} \mid \ L^S u = 0, |u|(x) \leq C(d^S(p,x)+1)^d\}$, where L^S is the Laplacian operator on (G,S), d^S is the distance function to some fixed $p \in G$. For the finitely generated abelian group $G = G_1 \oplus G_2 \cong \mathbb{Z}^m \bigoplus \oplus_{i=1}^{l} \mathbb{Z}_{q_i}$, we define the natural projection

$$\pi_{G_1} : \ G \to G_1,$$
$$x \mapsto \pi_{G_1}(x) = x_1,$$

for $x = x_1 + x_2$, $x_1 \in G_1$ and $x_2 \in G_2$. It is easy to see that $\pi_{G_1} S$ is a generating set of G_1 if S is a generating set of G.

Theorem 1.2 (Generalized Heilbronn's theorem)**.** *Let (G, S) be the Cayley graph of a finitely generated abelian group, $G = G_1 \oplus G_2 \cong \mathbb{Z}^m \bigoplus \oplus_{i=1}^{l} \mathbb{Z}_{q_i}$. Then*

$$H^d(G,S) = H^d(G_1, \pi_{G_1} S),$$

moreover any $f \in H^d(G,S)$ is a polynomial when it is restricted to $G_1 \cong \mathbb{Z}^m$, and it is constant on G_2 i.e. $f(x+w) = f(x)$, for $\forall x \in G$, $\forall w \in G_2$.

Nayar [41] proved a strong version of Heilbronn's theorem. We denote by $HM^d(G,S) := \mathrm{Span}\{u : G \to \mathbb{R} \mid \ L^S u = 0, u(x) \geq -C(d^S(p,x)+1)^d\}$ the linear span of one-sided bounded polynomial growth harmonic functions. It is trivial that $H^d(G,S) \subset HM^d(G,S)$. By the Harnack inequality (3.4), we give a geometric analysis proof of Nayar's theorem.

Theorem 1.3 (Generalized Nayar's theorem)**.** *Let (G, S) be the Cayley graph of a finitely generated abelian group, $G = G_1 \oplus G_2 \cong \mathbb{Z}^m \bigoplus \oplus_{i=1}^{l} \mathbb{Z}_{q_i}$. Then*

$$HM^d(G,S) = H^d(G,S) = H^d(G_1, \pi_{G_1} S),$$

moreover any $f \in H^d(G,S)$ is a polynomial when it is restricted to $G_1 \cong \mathbb{Z}^m$ and it is constant on G_2.

Thirdly, for discrete harmonic polynomials we calculate the precise dimension by linear algebra. In fact, instead of using the technical lemma (see [19, 23]) from difference equations as Heilbronn [22] did in the lattice case, we apply the dimension comparison argument with harmonic polynomials in $\mathbb{R}^n$. Conversely, our argument provides a proof of this difference equation lemma. Moreover, we can calculate the dimension of the space of polynomial growth harmonic functions on Cayley graphs of finitely generated abelian groups. It is surprising that the dimension of polynomial growth harmonic functions does not depend on the generating set S for the abelian group G. Actually, the graph structures

of two Cayley graphs of the abelian group G with two generating set S_1, S_2 can be quite different. While the Laplacian operator depends on the generating set, the dimension of polynomial growth harmonic functions does not. We denote by $HP^k(\mathbb{R}^m)$ the space of harmonic polynomials on $\mathbb{R}^m$ with degree less than or equal to k, $k \in \mathbb{N} \cup \{0\}$. It is well known (see [33]) that

$$\dim HP^k(\mathbb{R}^m) = \binom{m+k-1}{k} + \binom{m+k-2}{k-1}.$$

Theorem 1.4 (Dimension calculation)**.** *Let (G, S) be the Cayley graph of a finitely generated abelian group, $G = G_1 \oplus G_2 \cong \mathbb{Z}^m \bigoplus \oplus_{i=1}^{l} \mathbb{Z}_{q_i}$. Then*

$$\dim HM^d(G, S) = \dim H^d(G, S) = \dim HP^{[d]}(\mathbb{R}^m).$$

Since we do not use the representation formula, this method can be applied in more general settings. In fact, Bochner's formula is proved for Ricci flat graphs by Chung-Yau [9], but it is hard to get the volume control or the Poincaré inequality in that case. Lin-Yau [40] proved Bochner's formula for general graphs, but this does not lead to a version of Yau's gradient estimate that is nicely analogous to the case of Riemannian manifolds with nonnegative Ricci curvature. In the Cayley graph case, Kleiner [28] obtained the Poincaré inequality, but for non-abelian groups, Bochner's formula is unavailable (consider the free group case). For some special graphs which can be embedded into a surface with nonnegative sectional curvature in the sense of Alexandrov, Hua-Jost-Liu [26] proved the volume doubling property and the Poincaré inequality but Bochner's formula. It seems that Bochner's formula is sensitive to the local structure, but the volume growth property and the Poincaré inequality are not, c.f. [15]. Hence the abelian groups are very suitable candidates for the application of Bochner's formula and Yau's gradient estimate. In addition, Alexopoulos' theorem [3] is more general than ours, but it depends on the embedding of the nilpotent subgroup to simply connected Lie group. It seems hard to calculate the precise dimension of polynomial growth harmonic functions. We shall give the dimension estimate in an upcoming paper.

2 Preliminaries and Notations

Let G be an abelian group. It is called finitely generated if it has a finite generating set. In this paper, we assume that any generating set $S = \{s_1, s_2, \cdots, s_{2l}\}$ of G is symmetric, i.e. $S = -S$, or more precisely $s_i = -s_{i+l}, 1 \leq i \leq l$, but we do allow that elements of S, are repeated, that is possibly $s_i = s_j$ for some $i \neq j$. We also allow $0 \in S$. For any finitely generated abelian group G with a generating set S, we have the associated Cayley graph (V, E) for which $V = G$, and $xy \in E$, (or $x \sim y$) if $y - x \in S$, for $x, y \in V$. The duplicity of elements in S produces multiedges between vertices, and $0 \in S$ makes self-loops. The vertices x and y are called neighbors if $x \sim y$. The degree of a vertex x is the number of its neighbors. Note that all vertices in (G, S) have the same degree $\sharp S$. For the lattice $\mathbb{Z}^n$ with the standard generating set $S^0 = \{e_i\}_{i=1}^{2n}$, where $e_i = (0, \cdots, 1, \cdots, 0)$ is the i-th unit vector and $e_i = -e_{i+n}$, $1 \leq i \leq n$, we obtain the standard integer lattice in $\mathbb{R}^n$. This is the

object Heilbronn [22] and Nayar [41] studied. In this paper, we consider the discrete harmonic functions on Cayley graphs of G with arbitrary finite generating set S.

The Cayley graph of (G, S) is endowed with a natural metric, called the word metric (c.f. [6]). For any $x, y \in G$, the distance between them is defined as the length of the shortest path connecting x and y,

$$d^S(x,y) := \inf\{k \in \mathbb{N} \mid \exists\, x = x_0 \sim x_1 \sim \cdots \sim x_k = y\}.$$

Denote by $B_r^S(x) := \{y \in G \mid d^S(y,x) \leq r\}$ the closed geodesic ball centered at x of radius r ($r > 0$). The volume of $B_r^S(x)$ is $|B_r^S(x)| := \sharp G \cap B_r^S(x)$, i.e. the number of vertices contained in $B_r^S(x)$. For the subset $\Omega \subset G$, $d^S(x,\Omega) := \inf\{d^S(x,y) \mid y \in \Omega\}$ for any $x \in G$, $\partial\Omega := \{z \in G \mid d^S(z,\Omega) = 1\}$, and $\bar{\Omega} = \Omega \cup \partial\Omega$. For any function $f : \bar{\Omega} \to \mathbb{R}$, the discrete Laplacian operator is defined on Ω as ($x \in \Omega$)

$$L^S f(x) = \sum_{y \sim x} (f(y) - f(x)).$$

The function f is called discrete harmonic (subharmonic) on Ω if $L^S f(x) = 0$ (≥ 0), for any $x \in \Omega$. In the lattice case $(\mathbb{Z}^n, S^0)$, the Laplacian operator is

$$L^{S^0} f(x) = \sum_{i=1}^{2n} (f(x + e_i) - f(x)).$$

The gradient of f at $x \in \Omega$ is defined as $|\nabla^S f|(x) = \sqrt{\sum_{y \sim x}(f(y) - f(x))^2}$. We also need the partial difference operator (for $s \in S$ and $x \in \Omega$)

$$\delta_s f(x) := f(x + s) - f(x).$$

For Cayley graphs of (G, S_1) and (G, S_2), it is known (c.f. [29, 20]) that

$$C_1 d^{S_1}(x,y) \leq d^{S_2}(x,y) \leq C_2 d^{S_1}(x,y),$$

for any $x, y \in G$, where C_1 and C_2 depend only on S_1, S_2. They are bi-Lipschitz equivalent in the metric point of view. Hence for any $x \in G$, $r > 0$,

$$|B^{S_1}_{\frac{r}{C_2}}(x)| \leq |B_r^{S_2}(x)| \leq |B^{S_1}_{\frac{r}{C_1}}(x)|.$$

By the fundamental theorem of finitely generated abelian groups [47, 43], any finitely generated abelian group G is isomorphic to the direct sum $\mathbb{Z}^m \bigoplus \oplus_{i=1}^l \mathbb{Z}_{q_i}$, where $m, l \in \mathbb{N}$, $q_i = p_i^{a_i}$ for some prime number p_i and some $a_i \in \mathbb{N}$. Hence there exists a generating set $S^0 = \{e_1, \cdots, e_{2m}, w_1, \cdots, w_{2l}\}$ ($e_i = -e_{i+m}$, $w_j = -w_{j+l}$, for $1 \leq i \leq m$, $1 \leq j \leq l$) such that G is identified with $\mathbb{Z}^m \bigoplus \oplus_{i=1}^l \mathbb{Z}_{q_i}$, where $\{e_1 \cdots e_{2m}\}$ generates the torsion-free part $\mathbb{Z}^m$ and $\{w_1 \cdots w_{2l}\}$ generates the torsion part $\oplus_{i=1}^l \mathbb{Z}_{q_i}$. For (G, S^0) it is easy to see that $C_1(m, S^0) r^m \leq |B_r^{S^0}(x)| \leq C_2(m, S^0) r^m$, and $|B_{2r}^{S^0}(x)| \leq C_3(m, S^0)|B_r^{S^0}(x)|$, for any $x \in G$ and $r \geq 1$. Hence by the bi-Lipschitz equivalence, for the Cayley graph of (G, S) we have

$$\begin{aligned} C_1(m, S^0, S) r^m \leq |B_r^S(x)| \leq C_2(m, S^0, S) r^m, \\ |B_{2r}^S(x)| \leq C_3(m, S^0, S)|B_r^S(x)|. \end{aligned} \tag{2.1}$$

The volume growth property (2.1) is called the volume doubling property.

In the sequel, for simplicity we shall omit S from our notation, e.g., $B_r(x) := B_r^S(x)$, if it does not cause any confusion. Also harmonic functions on G mean discrete harmonic functions. And constants C may change from line to line.

3 Yau's Gradient Estimate

In this section, we prove Bochner's formula on finitely generated abelian groups and derive Yau's gradient estimate analogous to the case of Riemannian manifolds with nonnegative Ricci curvature.

Kleiner [28] proved the Poincaré inequality on Cayley graphs of finitely generated (not necessarily abelian) groups. Let (G, S) be the Cayley graph of the finitely generated abelian group

$$G \cong \mathbb{Z}^m \bigoplus \oplus_{i=1}^{l} \mathbb{Z}_{q_i}. \tag{3.1}$$

By the volume doubling property (2.1), we obtain the uniform Poincaré inequality.

Lemma 3.1 (Poincaré inequality, [28]). *Let (G, S) be the Cayley graph as (3.1). Then there exists a constant $C(m, S)$ such that for any $p \in G$, $R > 0$ and any function $f : B_{3R}(p) \to \mathbb{R}$ we have*

$$\sum_{x \in B_R(p)} (f(x) - f_{B_R})^2 \leq CR^2 \sum_{x,y \in B_{3R}(p); x \sim y} (f(x) - f(y))^2, \tag{3.2}$$

where $f_{B_R} = \frac{1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x)$.

Note that by the independent works of Delmotte [17] and Holopainen-Soardi [21] the Moser iteration can be carried out for harmonic functions on graphs satisfying the volume doubling property and the Poincaré inequality. The Caccioppoli inequality for harmonic functions was obtained for general graphs with bounded degree (c.f. [14, 39, 21]).

Lemma 3.2 (Caccioppoli inequality). *Let (G, S) be the Cayley graph as (3.1). For any harmonic function f on $B_{6R}(p) \subset G, R \geq 1,$ it holds that*

$$\sum_{x \in B_R(p)} |\nabla f|^2(x) \leq \frac{C}{R^2} \sum_{x \in B_{6R}(p)} f^2(x), \tag{3.3}$$

where $C = C(S)$.

The Moser iteration implies the Harnack inequality for positive harmonic functions.

Lemma 3.3 (Harnack inequality). *Let (G, S) be the Cayley graph as (3.1). Then there exist constants $C_1(m, S)$ and $C_2(m, S)$ such that for any $p \in G$, $R \geq 1$ and any positive harmonic function f on $B_{C_1 R}(p)$ we have*

$$\max_{B_R(p)} f \leq C_2 \min_{B_R(p)} f. \tag{3.4}$$

The mean value inequality follows from one part of Moser iteration (c.f. [14, 17, 21]).

Lemma 3.4 (Mean value inequality). *Let (G, S) be the Cayley graph as (3.1). Then there exists a constant $C_1(m, S)$ such that for any $p \in G$, $R > 0$ and any harmonic function f on $B_R(p)$ we have*

$$f^2(p) \leq \frac{C_1}{|B_R(p)|} \sum_{x \in B_R(p)} f^2(x). \tag{3.5}$$

The following Liouville theorem is a corollary of the Harnack inequality (3.4).

Lemma 3.5 (Liouville theorem)**.** *Let (G, S) be the Cayley graph as (3.1). Then any nonnegative harmonic function f on G is constant.*

Bochner's formula has been obtained on Ricci flat graphs (c.f. [9, 40]). For the case of Cayley graphs of finitely generated abelian groups, we present the proof here for the convenience of readers.

Lemma 3.6 (Bochner's formula)**.** *Let (G, S) be the Cayley graph as (3.1) and f be a harmonic function defined on $B_1(x)$, for $x \in G$. Then*

$$L^S|\nabla f|^2(x) \geq 0. \tag{3.6}$$

Proof. Let us denote $S = \{s_1, s_2, \cdots, s_{2l}\}$, where $s_i = -s_{i+l}, 1 \leq i \leq l$. Then

$$|\nabla f|^2(x) = \sum_{y\sim x}(f(y) - f(x))^2 = \sum_{i=1}^{2l} |\delta_{s_i} f|^2(x).$$

Without loss of generality, it suffices to prove

$$L^S|\delta_{s_1} f|^2(x) \geq 0.$$

$$\begin{aligned}
L^S|\delta_{s_1} f|^2(x) &= \sum_{y\sim x}(|\delta_{s_1} f|^2(y) - |\delta_{s_1} f|^2(x)) \\
&= \sum_{y\sim x}(f(y + s_1) - f(y))^2 - 2l|\delta_{s_1} f|^2(x) \\
&\geq \frac{1}{2l}[\sum_{y\sim x} f(y + s_1) - f(y)]^2 - 2l|\delta_{s_1} f|^2(x) && (3.7) \\
&= \frac{1}{2l}[\sum_{i=1}^{2l} f(x + s_i + s_1) - \sum_{y\sim x} f(y)]^2 - 2l|\delta_{s_1} f|^2(x) \\
&= \frac{1}{2l}[\sum_{z\sim(x+s_1)} f(z) - \sum_{y\sim x} f(y)]^2 - 2l|\delta_{s_1} f|^2(x) \\
&= \frac{1}{2l}[2lf(x + s_1) - 2lf(x)]^2 - 2l|\delta_{s_1} f|^2(x) && (3.8) \\
&= 0,
\end{aligned}$$

where we use the Hölder inequality in (3.7) and the harmonicity of f in (3.8). □

Combining Bochner's formula with previous results, we obtain Yau's gradient estimate for positive harmonic functions.

Proof of Theorem 1.1. We choose $C_1 = 7C_1'$, where $C_1' > 1$ is the constant $C_1(m, S)$ in Lemma 3.3. Bochner's formula (3.6) implies that $|\nabla f|^2$ is a subharmonic function on $B_{C_1 R}(x)$. Then the theorem follows from the mean value inequality (3.5) (since $\delta_{s_i} f$ is harmonic for any $s_i \in S$), the Caccioppoli inequality (3.3), the volume doubling property (2.1) and the Harnack inequality

(3.4),

$$
\begin{aligned}
|\nabla f|^2(x) &\leq \frac{C}{|B_R(x)|}\sum_{y\in B_R(x)}|\nabla f|^2(y)\\
&\leq \frac{C}{R^2|B_R(x)|}\sum_{y\in B_{6R}(x)}f^2(y)\\
&\leq \frac{C}{R^2|B_{6R}(x)|}\sum_{y\in B_{6R}(x)}f^2(y)\\
&\leq \frac{C}{R^2}f^2(x).
\end{aligned}
$$

□

Corollary 3.7. *Let (G,S) be the Cayley graph as (3.1). There exist constants $C_1(m,S)$ and $C_2(m,S)$ such that for any $x\in G$, $R\geq 1$ any harmonic function f on $B_{C_1R}(x)$ we have*

$$|\nabla f|(x)\leq \frac{C_2}{R}\operatorname*{osc}_{B_{C_1R}(x)} f, \tag{3.9}$$

where $\mathrm{osc}_{B_{C_1R}(x)} f=\max_{B_{C_1R}(x)} f-\min_{B_{C_1R}(x)} f$.

Proof. It suffices to choose $f-\min_{B_{C_1R}(x)} f+\epsilon$ (small ϵ) as the positive harmonic function in Theorem 1.1. □

4 Polynomial Growth Harmonic Functions are Polynomials

The fundamental theorem of finitely generated abelian groups implies that for any finitely generated abelian group G there exists a generating set $S^0=\{e_1,\cdots,e_{2m},w_1,\cdots,w_{2l}\}$ ($e_i=-e_{i+m}$, $w_j=-w_{j+l}$, for $1\leq i\leq m$, $1\leq j\leq l$) such that $G=G_1\oplus G_2$ is identified with $\mathbb{Z}^m\bigoplus\oplus_{i=1}^l\mathbb{Z}_{q_i}$, where $\{e_1\cdots e_{2m}\}$ generates the torsion-free part $G_1\cong\mathbb{Z}^m$ and $\{w_1\cdots w_{2l}\}$ generates the torsion part $G_2\cong\oplus_{i=1}^l\mathbb{Z}_{q_i}$. For some fixed $p\in G$, we denote by $H^d(G,S)=\{u:G\to\mathbb{R}\mid L^Su=0,\exists C\ s.t.\ |u(x)|\leq C(d^S(p,x)+1)^d\}$ the space of polynomial growth harmonic functions on G with growth rate less than or equal to d. For fixed S^0, G_1 is identified with $\mathbb{Z}^m$, then we denote by $P^d(\mathbb{Z}^m):=P^d(\mathbb{R}^m)$ the space of polynomials in $\mathbb{R}^m$ restricted to the lattice $\mathbb{Z}^m$ with degree less than or equal to d.

First, we consider the easy case that G is torsion-free, i.e. $G\cong\mathbb{Z}^m$. The following theorem generalizes the classical theorem of Heilbronn [22].

Theorem 4.1. *Let (G,S) be the Cayley graph of a torsion-free finitely generated abelian group, $G\cong\mathbb{Z}^m$. Then polynomial growth harmonic functions on G are polynomials, i.e.*

$$H^d(G,S)\subset P^d(\mathbb{Z}^m).$$

Proof. Let $S^0=\{e_1,\cdots,e_{2m}\}$ be the standard basis for $\mathbb{Z}^m$, $S=\{s_1,\cdots,s_{2l}\}$ be the generating set for the Cayley graph (G,S). Then

$$e_i=\sum_{k=1}^{2l}a_i^ks_k,$$

where $a_i^k \in \mathbb{Z}$ for $1 \le i \le 2m, 1 \le k \le 2l$. Let

$$\delta_i f(x) := \delta_{e_i} f(x) = f(x+e_i) - f(x),$$

for $1 \le i \le 2m$. Since G is abelian, it is easy to show that

$$L^S \delta_i f = \delta_i L^S f = 0,$$

for the harmonic function f.

We claim that $\delta_i f \in H^{d-1}(G,S)$ if $f \in H^d(G,S)$. Although for any $x \in G$, x and $x+e_i$ may not be neighbors in the Cayley graph (G,S), there exists a path from x to $x+e_i = x + \sum_{k=1}^{2l} a_i^k s_k$, i.e. $x = x_0 \sim x_1 \sim \cdots \sim x_t = x+e_i$, whose length is $t = \sum_{k=1}^{2l} |a_i^k| \le C(S^0,S)$. Note that $f \in H^d(G,S) \Longleftrightarrow \operatorname{osc}_{B_R(p)} f \le CR^d$ for some fixed $p \in G$ and sufficiently large $R \ge R_0$. For any $x \in G$, $1 \le i \le 2m$,

$$\begin{aligned}
|\delta_i f|(x) &= |f(x+e_i) - f(x)| \\
&\le |f(x+e_i) - f(x_{t-1})| + |f(x_{t-1}) - f(x_{t-2})| + \cdots + |f(x_1) - f(x)| \\
&\le \sum_{j=0}^{t-1} |\nabla f|(x_j) \\
&\le \frac{C}{R} \operatorname*{osc}_{B_{2R}(x)} f,
\end{aligned}$$

for $R \ge R_1(S^0,S)$, since $x_j \in B_C(x)$, for $0 \le j \le t-1$. The last inequality follows from Corollary 3.7. For any $x \in B_R(p)$, we have $B_{2R}(x) \subset B_{3R}(p)$. Then

$$\operatorname*{osc}_{B_R(p)} \delta_i f \le 2 \max_{x \in B_R(p)} |\delta_i f|(x) \le \frac{C}{R} \operatorname*{osc}_{B_{3R}(p)} f \le CR^{d-1},$$

for $R \ge R_1$. This proves the claim.

Hence by taking finitely many times partial differences, we obtain

$$\delta_1^{k_1} \delta_2^{k_2} \cdots \delta_{2m}^{k_{2m}} f = 0,$$

for any $k_1 + k_2 + \cdots + k_{2m} \ge [d]+1$ and $f \in H^d(G,S)$, where $[d]$ is the maximal integer not exceeding d. By the basic difference equation theory or Lemma 2.13 in Nayar [41], we conclude that f is a polynomial. $\square$

Then we can prove the generalized Heilbronn's theorem.

Proof of Theorem 1.2. It suffices to show that f is constant on G_2, i.e. $f(x+w) = f(x)$, for any $x \in G$, $w \in G_2$.

Let $S^0 = \{e_1, \cdots, e_{2m}, w_1, \cdots, w_{2l}\}$ ($e_i = -e_{i+m}$, $w_j = -w_{j+l}$, for $1 \le i \le m$, $1 \le j \le l$) such that $G = G_1 \oplus G_2$ is identified with $\mathbb{Z}^m \bigoplus \oplus_{i=1}^l \mathbb{Z}_{q_i}$, where $\{e_1 \cdots e_{2m}\}$ generates the torsion-free part $G_1 \cong \mathbb{Z}^m$ and $\{w_j, w_{j+l}\}$ generates $\mathbb{Z}_{q_i}$. Let $\delta_{w_j} f(x) := f(x+w_j) - f(x)$. The same argument as in the proof Theorem 4.1 implies that $\delta_{w_j} f \in H^{d-1}(G,S)$ if $f \in H^d(G,S)$. Then $\delta_{w_j}^{[d]+1} f \equiv 0$, for $f \in H^d(G,S)$. For fixed $x \in G$, lifting $\mathbb{Z}_{q_j}$ to $\mathbb{Z}$, we obtain that f is a polynomial on $\mathbb{Z}$. Since $f(x + (r+kq_j)w_j) = f(x + rw_j)$ for any $k, r \in \mathbb{Z}$, the periodic polynomial f must be constant, i.e. $f(x+rw_j) = f(x)$, for any $r \in \mathbb{Z}$. Since this is true for any $1 \le j \le l$, we obtain $f(x+w) = f(x)$, for any $x \in G$, $w \in G_2$. Then it is easy to see that $L^{\pi_{G_1} S} f(x_1) = 0$ for any $x_1 \in G_1$ if $L^S f(x) = 0$ for any $x \in G$. Hence $H^d(G,S) = H^d(G_1, \pi_{G_1} S) \subset P^{[d]}(\mathbb{Z}^m)$. $\square$

By the Harnack inequality, we reprove Nayar's theorem.

Proof of Theorem 1.3. It suffices to show $HM^d(G,S) \subset H^d(G,S)$. Without loss of generality, for $f \in HM^d(G,S)$ satisfying $f(x) \geq -C(d(p,x)+1)^d$, we need to prove that $f(x) \leq C(d(p,x)+1)^d$, for some C. For simplicity, we assume $f(p) = 0$. Let C_1 be the constant for the Harnack inequality in the Lemma 3.3. Then for any $x \in B_R(p)$, $R > 0$, it is easy to see that $B_{C_1R}(x) \subset B_{(C_1+1)R}(p)$. Moreover

$$f(y) \geq -C(d(p,y)+1)^d \geq -C((C_1+1)R+1)^d \geq -CR^d,$$

for $y \in B_{(C_1+1)R}(p)$, $R \geq 1$. That is $f(y) + CR^d \geq 0$ on $B_{C_1R}(x)$. The Harnack inequality (3.4) implies that

$$f(x) + CR^d \leq C(f(p) + CR^d) = C_2R^d.$$

Then we have

$$f(x) \leq CR^d,$$

for $x \in B_R(p)$, $R \geq 1$. Hence there exists a constant C such that $f(x) \leq C(d(p,x)+1)^d$. $\square$

5 Calculating the Dimension

For calculating the dimension, by Theorem 1.2, it suffices to consider harmonic polynomials on a torsion-free finitely generated abelian group. Let (G,S) be the Cayley graph of $G \cong \mathbb{Z}^m$. There exists a generating set $S^0 = \{e_1, \cdots, e_{2n}\}$ such that G is identified with $\mathbb{Z}^m$ in $\mathbb{R}^m$. For $k \in \mathbb{N} \cup \{0\}$, we denote by $P^k(\mathbb{R}^m)$ the space of polynomials on $\mathbb{R}^m$ of degree less than or equal to k, by $P^k_m := P^k(\mathbb{Z}^m) := \{u : \mathbb{Z}^m \to \mathbb{R} \mid u|_{\mathbb{Z}^m} = f|_{\mathbb{Z}^m}, f \in P^k(\mathbb{R}^m)\}$ the space of the restriction of polynomials on $\mathbb{R}^m$ to $\mathbb{Z}^m$ of degree less than or equal to k. We denote by $R^k_m := HP^k(\mathbb{R}^m)$ the space of harmonic polynomials on $\mathbb{R}^m$ ($\Delta u = 0$) of degree less than or equal to k. For the Cayley graph (G,S) which is identified with $(\mathbb{Z}^m, S)$, we set $D^k_{S,m} := HP^k(\mathbb{Z}^m, S) := \{u \in P^k(\mathbb{Z}^m) \mid L^S u = 0\}$. In order to calculate the dimension of discrete harmonic polynomials, we make the dimension comparison between $D^k_{S,m}$ and R^k_m. It is well known for harmonic polynomials on $\mathbb{R}^m$ (see [33]) that

$$\begin{aligned}
\dim R^k_m &= \binom{m+k-1}{k} + \binom{m+k-2}{k-1}, && (k \geq 1,\ \dim R^0_m = 1)\\
\dim R^k_m &= \dim R^k_{m-1} + \dim R^{k-1}_m, && (k \geq 1)\\
\dim P^k_m &= \dim P^k(\mathbb{R}^m) = \sum_{i=0}^{k} \binom{m+i-1}{i}, && (k \geq 0)\\
\dim P^k_m &= \dim R^k_m + \dim P^{k-2}_m. && (k \geq 2)
\end{aligned}$$

Lemma 5.1.

$$\dim D^k_{S,1} = \dim R^k_1 = 2, \tag{5.1}$$

for any S, $k \geq 1$.

Proof. Let $S = \{a_i\}_{i=1}^{2l}$, $a_i \in \mathbb{Z}$, and $a_i = -a_{i+l}$ for $1 \leq i \leq l$. Since S is a generating set, at least one $a_i \neq 0$. For any $f \in D^k_{S,1}$, $f = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0$, where $b_0, b_1, \cdots, b_k \in \mathbb{R}$. By Taylor expansion, the difference equation $L^S f = 0$ reads

$$\sum_{n=1}^{\infty}\sum_{i=1}^{l} 2\frac{a_i^{2n}}{(2n)!} f^{(2n)}(x) = 0.$$

Comparing the degree of polynomials, we have

$$(b_k x^k)'' = 0,$$

for $x \in \mathbb{Z}$. Hence, $k \leq 1$, i.e. f is linear. □

Lemma 5.2.

$$\dim D^k_{S,m} \leq \dim R^k_m, \tag{5.2}$$

for any S, $k \geq 0, m \geq 1$.

Proof. It is easy to see that $D^0_{S,m} = R^0_m = const$. We apply an induction argument on k. It suffices to prove (5.2) for $k = l$ if (5.2) is true for all $k \leq l-1$. We pick $e_1 \in S^0$ and define $\delta_1 f(x) = f(x+e_1) - f(x)$, for any function f on G. Since G is abelian, $\delta_1 L^S = L^S \delta_1$, then δ_1 is a well defined linear operator,

$$\delta_1 : D^k_{S,m} \to D^{k-1}_{S,m}.$$

By linear algebra,

$$\dim D^k_{S,m} = \dim \ker \delta_1 + \dim \mathrm{im}\ \delta_1, \tag{5.3}$$

where $\ker \delta_1$ and im δ_1 are Kernel and Image of δ_1. There is a natural projection $P : \mathbb{Z}^m \to \mathbb{Z}^{m-1}$, for any $x = (x_1, x_2, \cdots, x_m) \in \mathbb{Z}^m$, $Px = x'$, where $x' = (x_2, \cdots, x_m)$. For any generating set S for $\mathbb{Z}^m$, $S' = \{s' \mid s \in S\}$ is the generating set of $\mathbb{Z}^{m-1}$. Moreover, $S^{(t)} := P^t S$ is the generating set for $\mathbb{Z}^{m-t}$, $1 \leq t \leq m-1$.

For any $f \in \ker \delta_1$, i.e. $\delta_1 f = 0$, then $f(x_1, x_2, \cdots, x_m) = g(x_2, \cdots, x_m)$. Hence

$$\begin{aligned} 0 &= L^S f(x) = \sum_{s\in S}(f(x+s) - f(x)) \\ &= \sum_{s\in S}(g(x'+s') - g(x')) = L^{S'} g(x'). \end{aligned}$$

That is $\ker \delta_1 = D^k_{S',m-1}$. Hence (5.3) implies that

$$\dim D^k_{S,m} \leq \dim D^k_{S',m-1} + \dim D^{k-1}_{S,m},$$

for any S, $k \geq 1$, $m \geq 1$. Then it follows that

$$\begin{aligned}
\dim D^l_{S,m} &\leq \dim D^l_{S',m-1} + \dim D^{l-1}_{S,m} \\
&\leq \dim D^l_{S'',m-2} + \dim D^{l-1}_{S',m-1} + \dim D^{l-1}_{S,m} \\
&\leq \cdots\cdots \\
&\leq \dim D^l_{S^{(m-1)},1} + \sum_{i=2}^{m} \dim D^{l-1}_{S^{(m-i)},i} \\
&\leq 2 + \sum_{i=2}^{m} \dim R^{l-1}_i \\
&= \dim R^l_1 + \sum_{i=2}^{m} \dim R^{l-1}_i \\
&= \dim R^l_m,
\end{aligned}$$

where we use Lemma 5.1, the inductive assumption (5.2) for $k \leq l-1$ and some facts in $\mathbb{R}^n$.

□

Now we can prove the main Theorem 1.4.

Proof of Theorem 1.4. It suffices to show that

$$\dim D^k_{S,m} = \dim R^k_m,$$

for any S, $k \geq 0$ and $m \geq 1$. We may assume $k \geq 2$, otherwise it is trivial. By $S = -S$, the Laplacian operator L^S is a linear operator,

$$L^S : P^k_m \to P^{k-2}_m.$$

Since $\ker L^S = D^k_{S,m}$, we have

$$\begin{aligned}
\dim P^k_m &= \dim \ker L^S + \dim \mathrm{im}\ L^S \\
&\leq \dim D^k_{S,m} + \dim P^{k-2}_m \\
&\leq \dim R^k_m + \dim P^{k-2}_m \\
&= \dim P^k_m,
\end{aligned}$$

which follows from (5.2). Hence $\dim D^k_{S,m} = \dim R^k_m$. □

Remark 5.3. In the above theorems, all the inequalities for dimension comparison are equalities. Hence, we obtain that δ_1 and L^S are surjective linear operators. In fact, Heilbronn [22] used the technical lemma in difference equation theory (see [23, 19]) that $L^{S^0} : P^k_m \to P^{k-2}_m$ is surjective to calculate the dimension. Conversely, by dimension comparison, we obtain a more general difference equation lemma.

Corollary 5.4. *Let $(\mathbb{Z}^m, S)$ be the Cayley graph of $\mathbb{Z}^m$. Then*

$$\delta_1 : D^k_{S,m} \to D^{k-1}_{S,m},$$

and

$$L^S : P^k_m \to P^{k-2}_m$$

are surjective linear operators.

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